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Analysis of small-aspect-ratio lifting surfaces in ground effect

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A lifting surface of small aspect ratio is analysed for motion with constant forward velocity, parallel and in close proximity to a rigid plane surface of infinite extent. The gap flow beneath the lifting surface is represented by a simple nonlinear solution in the cross-flow plane, and appropriate conditions are imposed at leading and trailing edges. The transition between these two conditions depends on the kinematics of the gap flow as well as the planform geometry. For steady-state motion of a delta wing with sufficiently large angle of attack, the transition point is upstream of the tail. For oscillatory heaving motion of a delta wing the transition point is cyclic if the heave velocity is sufficiently large. Illustrative computations of the lift force are presented.

1. Introduction

The analysis of lifting surfaces moving in close proximity to a ground plane is of practical importance in the context of high-speed ground-transportation vehicles, and also with respect to the interaction between a ship hull and an adjacent canal wall or second ship. Possible biological applications exist as well, particularly with respect to prosthetic heart valves, which are similar to those occurring in nature but with a small gap between the valve leaves and the artery.

Restricting attention to a horizontal ground plane of infinite extent, and a lifting-surface geometry that is nearly parallel to this plane, a suitable inviscid model can be developed by extensions of classical thin-wing theory. If the clearance beneath the wing is small compared with the span and chord, the velocity field within the gap region is dominant and can be approximated by a governing differential equation analogous to the shallow-water approximation in water-wave theory. Widnall & Barrows (1970) exploit this simplification in a linearized steady-state analysis where the assumption is made that the angle of attack is small compared to the gap, but no restrictions are placed on the aspect ratio. Some unsteady extensions of the same theory are outlined by Barrows & Widnall (1970).

A similar small-gap approximation is employed by Yih (1974) to analyse the dynamics of a falling plate, and to explain the cushioning effect when, for example, one pane of glass falls upon another. Since there is no linearization of the vertical displacement with respect to the clearance, the results are applicable to extremely small gaps.

A more general study of the two-dimensional thin-wing problem has been made by

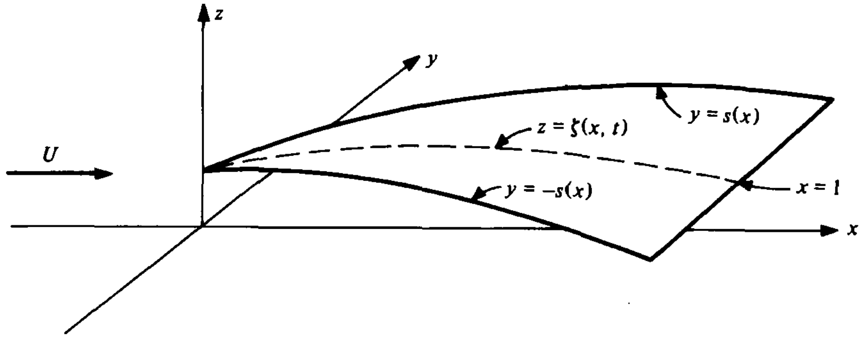

FIGURE 1. Definition of the co-ordinate system and lifting surface in close proximity to the ground-plane $z = 0$.

Tuck (1980), taking into account both unsteady motions of the wing and nonlinearity of the flow in the gap. Thus Tuck's results, like those of Yih (1974) for the non-lifting case, can be applied to the regime where the gap is very small and the ground effect is 'extreme'.

The present work is directed toward a solution of the unsteady three-dimensional thin-wing problem in the case where the aspect ratio is small. This assumption is more restrictive than the corresponding works of Barrows and Widnall, but the resulting simplification makes it possible to include nonlinear gap effects in an analogous manner to Tuck's two-dimensional study. The variation of clearance in the gap along the chord, due to the combined effects of camber, angle of attack, and vertical unsteady motions, is assumed to be comparable to the gap clearance itself. This nonlinear feature causes significant perturbations of the streaming flow in the gap region. In particular, the lateral deflection of streamlines is comparable to the span, and special attention is required to distinguish the appropriate portions of the planform boundary which correspond to leading and trailing edges. Transition between a leading and trailing edge will be shown to occur when the edge is tangent to the vector average of the velocity emerging from the gap and the adjacent velocity above the wing surface.

The analysis is performed for a wing of zero thickness, with small aspect ratio and small gap clearance. It is necessary to restrict the relative magnitudes of the two small parameters, such that the clearance is much less than the span. (In the complementary case ground effects only cause a small perturbation of the classical low-aspect-ratio flow.)

2. The boundary-value problem

Non-dimensional Cartesian co-ordinates (x, y, z) are defined as in figure 1, with $z = 0$ the plane of the ground, and with the local elevation of the wing prescribed by $z = \zeta(x, t)$ for $0 < x < 1$. The nose of the wing is situated on the (positive) z-axis, and directed towards a uniform streaming flow $(U, 0, 0)$. The wing surface is bounded by symmetric edges $y = \pm\, s(x)$, with the local semi-span $s(x)$ required to vanish at the nose and to vary slowly along the chord except for a possible abrupt trailing edge at the tail $x = 1$.

Inviscid incompressible flow is assumed, irrotational except for thin vortex sheets which are shed downstream from the trailing edges. At such edges the pressure must be continuous in accordance with the Kutta condition.

The fluid velocity field is expressed as the gradient of the potential $Ux+\phi(x,y,z,t)$, where

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{1}$$

throughout the fluid domain, and the perturbation potential ϕ is required to vanish at large distances from the wing and vortex sheets. The remaining boundary conditions are that

$$\phi_z=0 \quad \text{on} \quad z=0, \tag{2}$$

$$\phi_z=\zeta_t+(U+\phi_x)\,\zeta_x \quad \text{on} \quad z=\zeta(x,t), \quad |y|<s(x). \tag{3}$$

Following the geometrical assumptions outlined in § 1, it is assumed that

$$\zeta \ll s \ll 1$$

for all relevant values of (x,t). Co-ordinate stretching can be used to establish an inner region $(y,z)=O(s)$ where transverse gradients are dominant and, neglecting a factor $1+O(s^2)$, the perturbation potential is governed by the two-dimensional Laplace equation

$$\phi_{yy}+\phi_{zz}=0. \tag{4}$$

Similarly, the boundary condition (3) can be linearized in the form

$$\phi_z=\zeta_t+U\zeta_x\equiv D(\zeta) \quad \text{on} \quad z=\zeta(x,t), \quad |y|<s(x). \tag{5}$$

This linearization can be confirmed for the gap region *a posteriori*, with the error factor $1+O(\phi_x/U)=1+O(s^2)$.

In the usual procedure of matched asymptotic expansions, this inner problem must be complemented by an outer region where three-dimensional effects are significant, and the solutions in the two separate domains are matched ultimately in an appropriate overlap region. Matching is unnecessary for the leading-order solution of the present problem, however, and the inner solution may be derived directly from two-dimensional arguments. The higher-order role of the outer solution is described in § 5.

3. The leading-order solution

For sufficiently small values of the gap elevation ζ, the flow beneath the wing will dominate that in the exterior region. To confirm this statement, the perturbation field on the upper surface and elsewhere outside the gap is estimated as $O(\zeta)$ in accordance with classical thin-wing theory. By comparison, a (particular) solution for the flow in the gap beneath the wing is given by the potential

$$\phi=W(x,t)\,(z^2+s^2-y^2), \tag{6}$$

where

$$W(x,t)=\frac{D(\zeta)}{2\zeta}, \tag{7}$$

in accordance with the boundary condition (5). Note that (6) satisfies the boundary condition (2) and the two-dimensional Laplace equation (4). The additive constant Ws^2 has been introduced in anticipation that the potential (6) should vanish at the edges of the gap.

A general solution for the velocity potential in the gap region follows from (6) in the form

$$\phi = W(x,t)(z^2+s^2-y^2)+F(x,t), \tag{8}$$

where the complementary solution $F(x,t)$ is to be determined by imposing appropriate conditions at the edges of the gap $y = \pm s$.

Since the exterior perturbation is an order of magnitude smaller in ζ by comparison with (8), and local solutions which match the gap and exterior flows are significant only within a distance $O(\zeta)$ from the edges, the appropriate condition for the leading-edge portion of the gap is that

$$\phi = 0, \quad y = \pm s. \tag{9}$$

This condition is satisfied by (6), neglecting a term of order ζ^2, hence $F = 0$ in the leading-edge domain.

Trailing-edge conditions apply at the gap edges $y = \pm s$ in certain circumstances (which will be determined below). In this event, vortex sheets are shed, and convected downstream from the edges, and (9) must be replaced by the Kutta condition of continuous pressure. Since the perturbation field is of higher order above the wing, the pressure in the gap must therefore vanish at the trailing edges.

The pressure field in the gap can be obtained from Bernoulli's equation in the form

$$p/\rho = -\phi_t - U\phi_x - \tfrac{1}{2}\phi_y^2, \tag{10}$$

with the error a factor $1+O(\zeta^2/s^2, s^2)$. Substituting the general solution (8), and setting $z = 0$,

$$p/\rho = -D(W)(s^2-y^2) - WD(s^2) - 2W^2y^2 - D(F). \tag{11}$$

This pressure must vanish at the trailing edges, in accordance with the Kutta condition, and an equation for the complementary solution follows in the form

$$DF = -2UWss' - 2W^2s^2. \tag{12}$$

The solution of (12), which is continuous with the leading-edge potential at a transition point $x = x_*$, is readily obtained in the form

$$F(x,t) = -\int_{x_*}^{x} d\xi\{2W(\xi,\tau)s(\xi)s'(\xi) + [W(\xi,\tau)s(\xi)]^2/U\}. \tag{13}$$

Here the retarded time τ is defined by

$$\tau = t+(\xi-x)/U. \tag{14}$$

The transition point x_* is determined by noting that both the leading- and trailing-edge conditions are satisfied simultaneously if $F(x,t)$ and its derivative (12) both are equal to zero. This will occur in the leading-edge domain if the right-hand side of (12) vanishes, or if

$$s'/s = -W/U = -\tfrac{1}{2}(\zeta_t+U\zeta_x)/U\zeta. \tag{15}$$

Assuming x_* to be the first (or only) such point along the chord, leading-edge conditions are applicable upstream of this position.

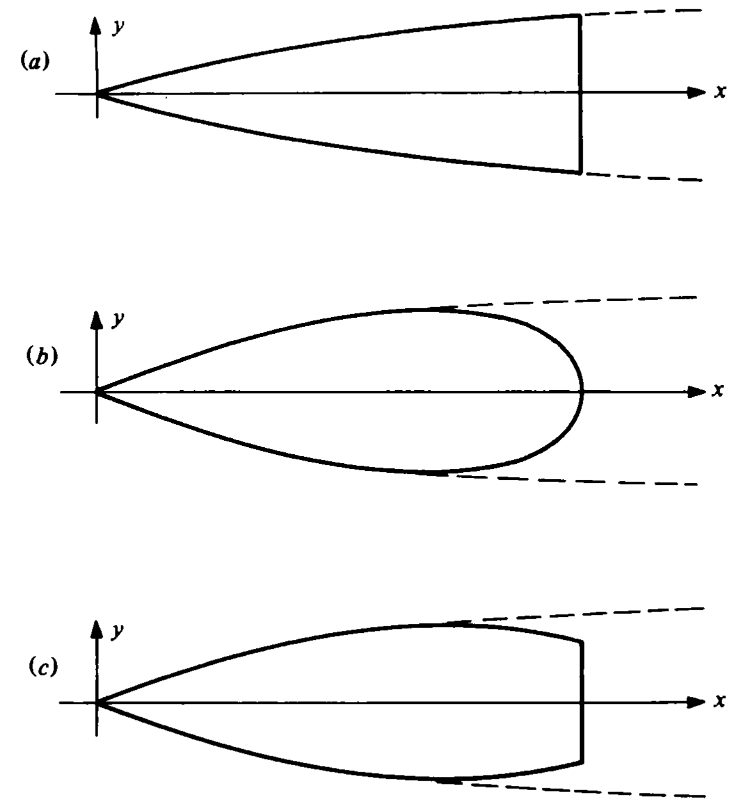

FIGURE 2. Planforms of lifting surfaces with (*a*) abrupt trailing edge, (*b*) swept trailing edge, and (*c*) combination of abrupt and swept trailing edges. Dashed lines denote typical positions of the outboard edges of the trailing vortex sheet.

Equation (15) can be interpreted from the kinematic standpoint by noting that the y-component of the velocity at the edge of the gap is equal to

$$\phi_y\,|_{y=\pm s} = -2Ws. \tag{16}$$

Thus (15) is satisfied if the slope of the streamline leaving the gap is *twice* the slope $s'(x)$ of the edge. Since the streamlines on the upper surface are unperturbed, this is equivalent to the statement that the mean flow at the edge is tangent to this edge. A leading-edge condition exists if the slope of this mean flow is algebraically less than $s'(x)$, but when the mean flow is directed outward vorticity is shed and convected in the same direction. Thus the appropriate distinction between the two types of edge conditions is that

$$\tfrac{1}{2}\phi_y\,|_{y=\pm s} \lessgtr Us'(x), \tag{17}$$

for a leading or trailing edge, respectively. Since $s(0) = 0$, leading-edge conditions are appropriate at the nose and upstream of the point x_*.

The point x_* may or may not exist upstream of the tail, depending on the wing geometry. If an abrupt edge exists at the tail, and if the edge slope s' is sufficiently small in relation to the normal velocity on the wing, a trailing edge will exist only at the tail, as depicted in figure 2 (*a*). Figure 2 (*b*) shows the opposite situation of a pointed

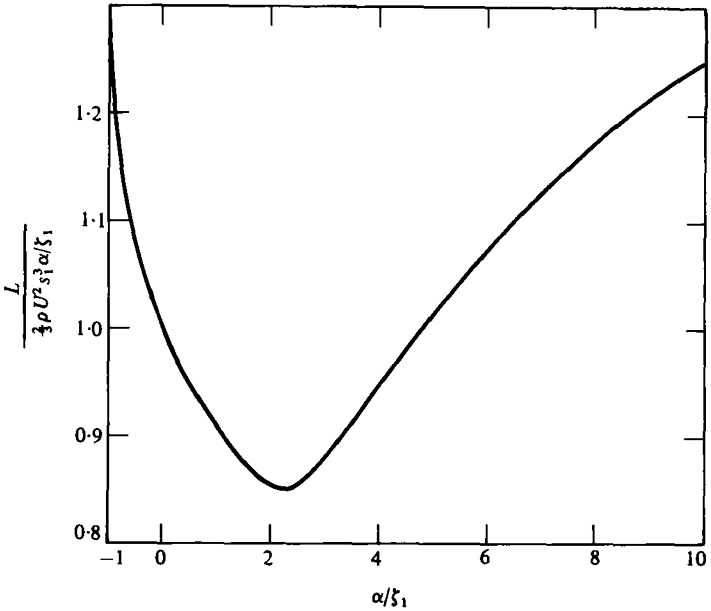

FIGURE 3. Normalized lift-slope coefficient for a planar delta wing with unit chord length, base span s_1, base elevation ζ_1, and angle of attack α.

tail with $s(1) = 0$, where trailing edges will exist along a portion of the chord. In the most general case both types of trailing edges may exist simultaneously, as shown in figure 2(*c*).

In certain cases the trailing vortex sheets will be 'resorbed' at a subsequent downstream leading edge, depending on the relative variation of the functions $s(x)$ and $W(x,t)$. It is straightforward to generalize the above results in this instance, subject to the requirement that the complementary solution $F(x,t)$ is continuous at each subsequent transition point. An analogous situation is described by Newman & Wu (1973) for the unsteady swimming motions of a slender fish.

4. The lift force and moment

The differential lift force acting on a transverse section of the wing is obtained to leading order by integrating the pressure across the gap,

$$\mathscr{L}(x,t) = -\rho \int_{-s}^{s} [\phi_t + U\phi_x + \tfrac{1}{2}\phi_y^2]\, dy. \tag{18}$$

Substituting (11) for the leading-edge region with $F = 0$ gives the corresponding result

$$\frac{1}{\rho}\mathscr{L} = -\tfrac{4}{3} D(Ws^3) - \tfrac{4}{3} W^2 s^3. \tag{19}$$

Similarly, from (11) and (12),

$$\frac{1}{\rho}\mathscr{L} = -\tfrac{4}{3} s^3 D(W) + \tfrac{8}{3} W^2 s^3 \tag{20}$$

for the trailing-edge region.

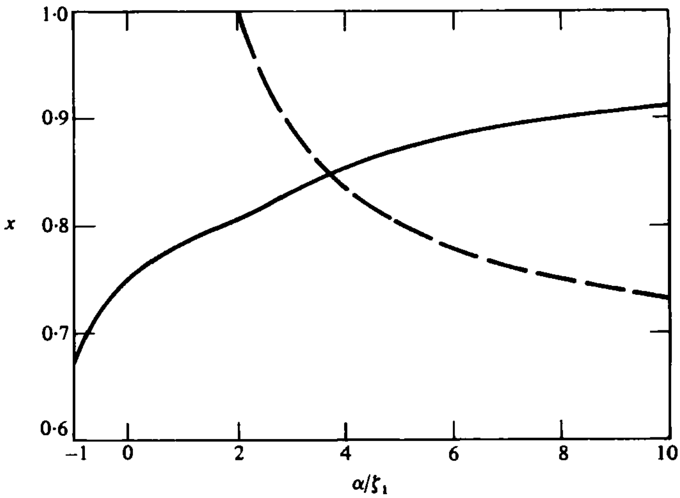

FIGURE 4. Centre of pressure (——) and transition point x_* between leading and trailing edges (– – –) for a planar delta wing.

The first terms on the right-hand sides of (19) and (20) correspond to the differential lift on a slender wing in an unbounded fluid, if W is replaced by the normal velocity $D\zeta$, and $\frac{4}{3}s^3$ is replaced by the local added-mass coefficient.

As a simple steady-state illustration we consider the motion of a planar delta wing with constant angle of attack α. Thus

$$s(x) = xs(1) \equiv xs_1, \tag{21}$$

$$\zeta(x) = \zeta(1)+\alpha(1-x) \equiv \zeta_1+\alpha(1-x). \tag{22}$$

For this case the transition position x_* is determined from (12),

$$x_* = 2\zeta(0)/3\alpha \equiv 2\zeta_0/3\alpha. \tag{23}$$

For $\alpha > \frac{2}{3}\zeta_0$, the transition position is upstream of the tail, approaching a limit point at $x = \frac{2}{3}$ when the tail elevation tends to zero. For negative values of the angle of attack, the trailing edge is confined to the abrupt tail.

The total lift force is obtained by substituting (21), (22) into (19), (20), and integrating over the chord. Figure 3 shows the normalized lift-slope coefficient

$$\frac{L}{\frac{2}{3}\rho U^2 s_1^3\,(\alpha/\zeta_1)}$$

for values of the ratio α/ζ_1 ranging from -1 (where the nose touches the ground) to $+10$. With this normalization, the lift-slope coefficient decreases from a maximum of 1·25 where the nose touches the ground to unity at zero angle of attack, and to a minimum value of about 0·85 just after the transition position moves forward from the tail (at $\alpha/\zeta_1 = 2$). Thereafter the lift slope increases markedly, and the classical term 'ram wing' is applicable. As $\alpha/\zeta_1 \to \infty$, the normalized lift-slope coefficient is asymptotic to a value of 2·0.

Figure 4 shows the corresponding results for the centre of pressure, which moves downstream with increasing α/ζ_1, from a position $\frac{2}{3}$ of the chord length aft of the nose

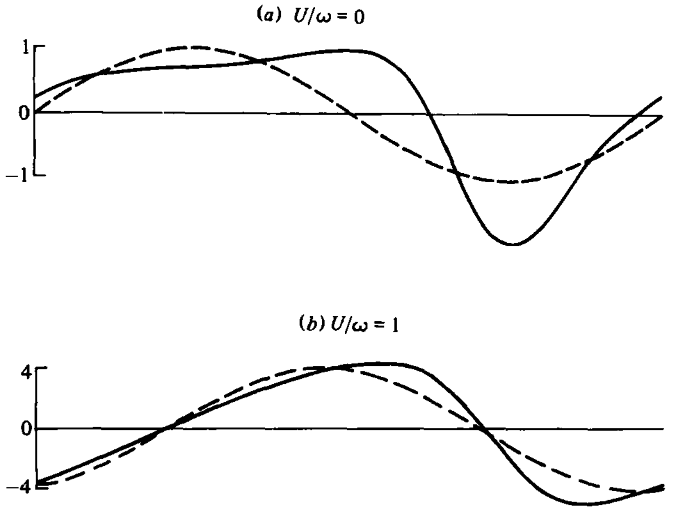

FIGURE 5. Unsteady lift coefficient (26) *vs.* ωt in the interval $(0, 2\pi)$. The solid curves are for a sinusoidal heave amplitude equal to half the mean elevation and the broken curves are the corresponding linearized results.

when the nose is touching the ground, to $\frac{3}{4}$ at zero angle of attack, and ultimately to a limiting position at the tail as $\alpha/\zeta_1 \to \infty$. The transition position x_* is shown also in figure 4 for values of the angle-of-attack ratio where this point is forward of the tail. When α/ζ_1 exceeds a value of approximately 3·7, the centre of pressure is in the trailing-edge region.

These steady-state results contrast with the corresponding situation in an unbounded fluid, where the differential lift force vanishes in the trailing-edge region. In the ground-effect problem, for large values of the ratio α/ζ_1, the parameter W increases inversely with the diminishing gap clearance near the tail, and the pressure (11) associated with the leading-edge potential is dominated by the negative term $-2W^2y^2$. In the trailing-edge regime this negative pressure at the edges is offset by an equal and opposite constant, in accordance with the Kutta condition, and a large positive pressure occurs inboard. Thus the total lift force increases with increasing α/ζ_1, and the centre of pressure moves toward the tail.

The simplest unsteady problem is that where the elevation is a function only of time, $\zeta = \zeta(t)$. In this case leading- or trailing-edge conditions occur respectively according as

$$s'/s \gtrless -\zeta'/2U\zeta, \tag{24}$$

where ζ' denotes the time derivative. For a falling delta wing, if ζ'/ζ is sufficiently large and negative, trailing-edge conditions will predominate. The corresponding differential lift force (20) reduces to a form identical to that obtained by Yih (1974) for a falling two-dimensional flat plate. In this situation there are no lifting effects associated with the stream velocity U.

To illustrate the unsteady problem we consider sinusoidal heaving motion

$$\zeta(t) = a + b \sin \omega t \tag{25}$$

of the delta planform defined by (21). In this case the pressure can be integrated to give the lift coefficient in the form

$$\frac{L}{\frac{1}{6}\rho s_1^3\,\omega^2 B} = \frac{\cos\omega t}{(1+B\sin\omega t)^2} - \frac{4U}{\omega}x_* \frac{\cos\omega^2 t}{(1+B\sin\omega t)} + B\frac{\cos^2 t}{(1+B\sin\omega t)^2}(1-\tfrac{3}{2}x_*^4). \tag{26}$$

Here $B = b/a$ is the ratio of the heave amplitude to the mean elevation. In (26), the parameter x_* is defined as 1 when the transition point is downstream of the trailing edge or, more generally,

$$\frac{1}{x_*} = \max\left\{1,\ -\frac{\omega B}{2U}\frac{\cos\omega t}{1+B\sin\omega t}\right\}. \tag{27}$$

For $U = 0$, $x_* = (0, 1)$ according as $\cos\omega t$ is negative or positive, respectively.

The lift coefficient (26) is shown in figure 5, as a periodic function of time, for the linear limit $B = 0$ and for $B = 0{\cdot}5$. Nonlinear effects are most apparent for $U/\omega = 0$. There is a positive average value of the lift coefficient over the cycle equal in this case to $0{\cdot}0774$. For $U/\omega = 1$, the forward velocity is sufficiently large to maintain the transition point at the tail; nonlinear effects are less apparent in this case, but the time-averaged lift coefficient is negative, and equal to $-0{\cdot}1547$ for $B = 0{\cdot}5$.

5. Discussion and conclusions

The relatively simple solution described above is valid to leading order in the small clearance ζ. A consistent higher-order solution can be derived from the method of matched asymptotics, in an analogous manner to that indicated for the two-dimensional problem by Widnall & Barrows (1970) and by Tuck (1980). The essential ingredients of such an extension for the wing of small aspect ratio include (1) a continuation of the gap flow downstream to account for the wake beneath the trailing-vortex sheet, (2) local edge solutions valid near $y = \pm s(x)$, and (3) an external solution valid throughout the remainder of the flow field including the upper surface of the wing.

The wake solution downstream of the gap involves a straightforward extension of (8), with continuity of the pressure at $x = 1$. The local edge solutions are described by Widnall & Barrows (1970) and can be decomposed into a particular solution with finite pressure at the edge, and a homogeneous source-like flow with the usual square-root edge singularity. Both components are required in general to treat the leading and trailing edges. The external solution consists of a planar distribution of sources on the wing and wake, including discrete sources along the leading edge. In the low-aspect-ratio approximation this external solution takes the form of the longitudinal 'thickness' problem in slender-body theory, including two-dimensional logarithmic terms plus a three-dimensional interaction from the remainder of the body and the wake. The latter will introduce significant three-dimensional effects in the higher-order solution.

Another possible extension of the theory is to include thickness of the wing. This has no effect on the leading-order solution, provided the clearance $\zeta(x,t)$ is defined as the elevation of the lower surface of the wing. In the higher-order theory, the upper surface should be used to develop the external solution, and thickness will affect the local edge solutions if the radius of curvature at the edges is comparable with the clearance.

Leading-edge separation must be anticipated unless the angle of attack is limited relative to the span and thickness. In principle, the restriction $\zeta \ll s$ should preclude leading-edge separation if a finite radius exists at the leading edges. In practice, however, transition between the leading and trailing edges may depend as much on the local radius at the edge as on the global kinematics of the vortex sheets.

The illustrative example of a planar delta wing considered in § 4 can be generalized, and a wide variety of planforms, camber distributions, and unsteady motions may be analysed from the relatively simple leading-order solution in § 3. Multiple transitions can occur between leading- and trailing-edge regions if the span $s(x)$ and elevation $\zeta(x,t)$ are non-monotonic.

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